Intermediate Report Nr. 1

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1 How to estimate $|g(t)|$ out of ED samples y_i?

1.1 Noise-free scenario

Let $f(t)$ be the raised-cosine (RC) filtered channel impulse response (CIR) modulated by a carrier with frequency $\omega_c = 2\pi f_c$, and let $g(t)$ be its according complex baseband equivalent[1]. Thus,

$$f(t) \quad = \quad \Re\left\{g(t)e^{\jmath\omega_c t}\right\} = \Re\left\{g(t)\right\}\cos\omega_c t - \Im\left\{g(t)\right\}\sin\omega_c t =: g_a(t)\cos\omega_c t - g_b(t)\sin\omega_c t \tag{1}$$

The output of the ED can now be calculated as

$$y_i \quad = \quad \int_{iT_s-\frac{T_i}{2}}^{iT_s+\frac{T_i}{2}} f^2(t)dt = \int_{iT_s-\frac{T_i}{2}}^{iT_s+\frac{T_i}{2}} \left(g_a(t)\cos\omega_c t - g_b(t)\sin\omega_c t\right)^2 dt \tag{2}$$

$$= \quad \int_{iT_s-\frac{T_i}{2}}^{iT_s+\frac{T_i}{2}} \left(g_a^2(t)\cos^2\omega_c t - 2g_a(t)g_b(t)\cos\omega_c t\sin\omega_c t + g_b^2(t)\sin^2\omega_c t\right)dt, \tag{3}$$

where T_i is the integration interval and T_s is the sampling interval of the ED. Assuming that $f_cT_i >> 1$ we can approximate y_i by

$$y_i \quad \approx \quad \int_{iT_s-\frac{T_i}{2}}^{iT_s+\frac{T_i}{2}} \left(g_a^2(t)\underbrace{\mathrm{E}\left\{\cos^2\omega_c t\right\}}_{0.5} - 2g_a(t)g_b(t)\underbrace{\mathrm{E}\left\{\cos\omega_c t\sin\omega_c t\right\}}_{0} + g_b^2(t)\underbrace{\mathrm{E}\left\{\sin^2\omega_c t\right\}}_{0.5}\right)dt \tag{4}$$

$$= \quad \int_{iT_s-\frac{T_i}{2}}^{iT_s+\frac{T_i}{2}} \frac{g_a^2(t)+g_b^2(t)}{2}dt = \int_{iT_s-\frac{T_i}{2}}^{iT_s+\frac{T_i}{2}} \frac{|g(t)|^2}{2}dt \tag{5}$$

The fact that this approximation holds was simulated in `testapproxgf.m`, where I iterated over all defined channel center frequencies and over integration periods $T_i = \{1, 2, 4\}$ ns. The results where then averaged over 50 channel realizations. Tab. 1 shows the approximation error Q_2, which is defined as

$$Q_2 \quad = \quad 10\log\left(\frac{\sum_i |y_i - \hat{y}_i|}{\sum_i |y_i|}\right), \tag{6}$$

where y_i is obtained by integrating over $f(t)$, whereas $\hat{y}_i$ is obtained by integrating over $\frac{|g(t)|}{2}$. As expected, the error reduces with increasing integration periods T_i and increasing center frequency f_c. Channels 4, 7, 11, and 15 show bigger errors due to the fact that these channels use shorter pulse durations (higher pulse repetition frequencies). Channel 0 is the baseband channel, that is f_c is equal to 499.2 MHz, which happes to be the PRF.

[1] This approach now swaps root raised-cosine (RRC) filtering and modulation, that is instead of RRC-modulate-RRC we have RC-modulate. This is something I still would like to analyze if this is valid, especially if the RRC receiver filter really outputs either the envelope of $f(t)$ (for high ω_c) or directly $f(t)$ (for low ω_c).

Using Urkowitz' approximation [1]

$$\int_0^{T_i} s^2(t)dt \approx \frac{1}{2W}\sum_{i=1}^{2T_iW} s^2(\frac{i}{2W}), \tag{7}$$

where W denotes the bandwidth of $s(t)$ we can simplify the integral to a sampling. $g(t)$, as it can be shown, is bandlimited with $W = \frac{1+\beta}{2T}$, where β is the roll-off factor and T the duration of the RC-filtered pulse. In order to sample only once per integration interval we have to set

$$T_i = \frac{1}{2W} = \frac{1}{2\frac{1+\beta}{2T}} = \frac{T}{1+\beta}. \tag{8}$$

Thus, we can say that the ED output y_i is approximated by

$$y_i \approx \int_{iT_s-\frac{T_i}{2}}^{iT_s+\frac{T_i}{2}} \frac{|g(t)|^2}{2}dt \approx \frac{1}{2W}\frac{|g(iT_s)|^2}{2} = \frac{T}{1+\beta}\frac{|g(iT_s)|^2}{2} \tag{9}$$

or, equivalently,

$$|g(iT_s)|^2 \approx \frac{2(1+\beta)}{T}y_i \tag{10}$$

In order to fully restore $|g(t)|$ the Nyquist theoreme has to be fulfilled, meaning that $2T_sW < 1$, therefore $T_s < T_i$. Then, and only then, we can say

$$|g(t)| \approx \sum_i |g(iT_s)|\text{sinc}\left(\frac{t-iT_s}{T_s}\right) \tag{11}$$

1.2 Noise-only scenario

In this case let us assume that the input $f(t)$ to the ED is RRC-filtered Gaussian noise modulated with a carrier frequency of ω_c, and let $n(t)$ be its zero-mean, complex Gaussian equivalent baseband signal, which is a lowpass process with a bandwidth of W:

$$f(t) = \Re\left\{n(t)e^{\jmath\omega_c t}\right\} = n_a(t)\cos\omega_c t - n_b(t)\sin\omega_c t \tag{12}$$

In accordance with [1] we can say that for sufficiently large T_i:

$$\int_{iT_s-\frac{T_i}{2}}^{iT_s+\frac{T_i}{2}} (n_a(t)\cos\omega_c t - n_b(t)\sin\omega_c t)\,dt \approx \int_{iT_s-\frac{T_i}{2}}^{iT_s+\frac{T_i}{2}} \frac{n_a^2(t)+n_b^2(t)}{2}dt \tag{13}$$

Since the RRC filter is only the *root* of the RC filter, we can assume that the bandwidths of these filters are identical, at least with respect to required sampling rates. Therefore, according to [1] we can state:

$$\int_{iT_s-\frac{T_i}{2}}^{iT_s+\frac{T_i}{2}} \frac{n_a^2(t)+n_b^2(t)}{2}dt \approx \frac{1}{2W}\sum_{i=1}^{2T_iW} \frac{n_a^2(\frac{i}{2W})+n_b^2(\frac{i}{2W})}{2} \tag{14}$$

Let us now assume that each component $n_a(t)$, $n_b(t)$ has a double-sided power spectral density (PSD) of $\frac{N_0}{2}$. Thus, the variance of the bandlimited noise signal is equal to $\sigma^2 = \frac{N_0}{2}2W = N_0W$. Taking this into account we obtain the following result:

$$\int_{iT_s-\frac{T_i}{2}}^{iT_s+\frac{T_i}{2}} \frac{n_a^2(t)+n_b^2(t)}{2}dt \approx \frac{1}{2W}\frac{n_a^2(\frac{i}{2W})+n_b^2(\frac{i}{2W})}{2} \tag{15}$$

$$\mathrm{E}\left\{\int_{iT_s-\frac{T_i}{2}}^{iT_s+\frac{T_i}{2}} \frac{n_a^2(t)+n_b^2(t)}{2}dt\right\} \approx \mathrm{E}\left\{\frac{1}{2W}\frac{n_a^2(\frac{i}{2W})+n_b^2(\frac{i}{2W})}{2}\right\} \tag{16}$$

$$= \frac{1}{2W}\mathrm{E}\left\{\frac{n_a^2(\frac{i}{2W})+n_b^2(\frac{i}{2W})}{2}\right\} \tag{17}$$

$$= \frac{1}{4W}\left(\mathrm{E}\left\{n_a^2(\frac{i}{2W})\right\}+\mathrm{E}\left\{n_b^2(\frac{i}{2W})\right\}\right) \tag{18}$$

$$= \frac{1}{4W}2\sigma^2 = 2N_0W\frac{1}{4W} = \frac{N_0}{2} \tag{19}$$

That this relation holds was shown experimentally in `testnoise.m`, whereas I would like to present more accurate results at a later time.

1.3 Noise and pulses

Combining the results from the previous two subsection, we can say that eventually we can expect that if

$$f(t) = \Re\left\{(g(t)+n(t))\,e^{\jmath\omega_c t}\right\} = (g_a(t)+n_a(t))\cos\omega_c t - (g_b(t)+n_b(t))\sin\omega_c t \tag{20}$$

we can compute the ED output y_i to:

$$y_i = \int_{iT_s-\frac{T_i}{2}}^{iT_s+\frac{T_i}{2}} f^2(t)dt \approx \frac{1}{2}\int_{iT_s-\frac{T_i}{2}}^{iT_s+\frac{T_i}{2}} (g_a(t)+n_a(t))^2+(g_b(t)+n_b(t))^2\,dt \tag{21}$$

$$= \frac{1}{2}\int_{iT_s-\frac{T_i}{2}}^{iT_s+\frac{T_i}{2}} |g(t)|^2+2g_a(t)n_a(t)+2g_b(t)n_b(t)+n_a^2(t)+n_b^2(t)dt \tag{22}$$

If we can assume that the noise signal is independent from the complex equivalent baseband CIR, we can compute the expected ED output:

$$\mathrm{E}\{y_i\} = \frac{1}{2}\int_{iT_s-\frac{T_i}{2}}^{iT_s+\frac{T_i}{2}} |g(t)|^2+\underbrace{\mathrm{E}\{2g_a(t)n_a(t)+2g_b(t)n_b(t)\}}_{0}+\mathrm{E}\left\{n_a^2(t)+n_b^2(t)\right\}dt \tag{23}$$

$$= \frac{1}{2}\int_{iT_s-\frac{T_i}{2}}^{iT_s+\frac{T_i}{2}} |g(t)|^2dt+\frac{1}{2}\int_{iT_s-\frac{T_i}{2}}^{iT_s+\frac{T_i}{2}} \mathrm{E}\left\{n_a^2(t)+n_b^2(t)\right\}dt \tag{24}$$

$$= \frac{1}{2}\int_{iT_s-\frac{T_i}{2}}^{iT_s+\frac{T_i}{2}} |g(t)|^2dt+\underbrace{\mathrm{E}\left\{\frac{1}{2}\int_{iT_s-\frac{T_i}{2}}^{iT_s+\frac{T_i}{2}} n_a^2(t)+n_b^2(t)\right\}dt}_{\frac{N_0}{2}} \tag{25}$$

$$= \frac{T}{1+\beta}\frac{|g(iT_s)|^2}{2}+\frac{N_0}{2} \tag{26}$$

References

[1] H. Urkowitz, "Energy detection of unknown deterministic signals," *Proceedings of the IEEE*, vol. 55, no. 4, pp. 523–531, April 1967.

Channel	$T_i = 1$ ns	$T_i = 2$ ns	$T_i = 4$ ns
0	-8.8679	-10.8324	-14.4363
1	-17.5851	-19.4822	-22.8703
2	-18.0563	-20.0375	-23.3192
3	-18.4409	-20.4495	-23.5822
4	-11.7113	-15.0096	-19.0712
5	-19.2925	-21.2875	-24.4124
6	-19.3739	-21.3941	-24.5375
7	-15.3554	-18.5553	-21.0280
8	-19.4163	-21.4675	-24.7160
9	-19.4189	-21.4937	-24.8704
10	-19.3846	-21.4178	-24.8242
11	-14.9386	-17.8392	-21.8113
12	-19.3001	-21.2549	-24.7425
13	-19.1953	-21.0878	-24.5631
14	-19.0437	-20.9195	-24.3747
15	-15.6152	-18.5142	-22.4674

Table 1: Approximation of $f(t)$ by $|g(t)|$ - Error Q_2 in dB