Intermediate Report Nr. 4

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1 Distribution of Noise/Noise characteristics after correlating and averaging

Starting from the knowledge that both the continuous, bandlimited within $[-W, W]$, noise components $n_a(t)$ and $n_b(t)$, as well as the critically sampled noise terms $n_a(\frac{i}{2W}) = n_a[i]$ and $n_b(\frac{i}{2W}) = n_b[i]$ are Gaussian distributed with zero-mean and variance $\sigma^2 = N_0 W$, we can derive the following, assuming that the PSD is flat within the limited band:

1.1 After Energy Detection

Knowing Urkowitz' approximation holds with

$$y[i] = \int_{iT_s - \frac{T_i}{2}}^{iT_s + \frac{T_i}{2}} |n(t)|^2 dt \quad \approx \quad \frac{1}{2W} |n[i]|^2 \tag{1}$$

with

$$n(t) = n_a(t) + \jmath n_b(t) \tag{2}$$

$$n[i] = n_a[i] + \jmath n_b[i] \tag{3}$$

we can calculate the noise characteristics after the ED using

$$n_{a,b}[i] \sim N\left(0, \sigma^2\right) \tag{4}$$

$$|n[i]|^2 \sim \Gamma\left(1, 2\sigma^2\right) \tag{5}$$

$$\tag{6}$$

and the fact that the cumulants of the Gamma distribution are calculated according to

$$\Gamma(\alpha, \theta) \rightarrow c_k = \alpha, \theta^k (k-1)! \tag{7}$$

$$c_1 = \mathrm{E}\{.\} = \alpha\theta \tag{8}$$

$$c_2 = \mathrm{Var}\{.\} = \alpha\theta^2. \tag{9}$$

Therefore, if $y[i]$ denote the ED output samples, we can state that

$$2Wy_i \sim \Gamma\left(1, 2\sigma^2\right) \tag{10}$$

$$\mathrm{E}\{2Wy[i]\} = 2\sigma^2 \rightarrow \mathrm{E}\{y[i]\} = \frac{1}{2W} 2\sigma^2 = \frac{N_0 W}{W} = N_0 \tag{11}$$

$$\mathrm{Var}\{2Wy[i]\} = 4\sigma^4 \rightarrow \mathrm{Var}\{y[i]\} = \frac{1}{4W^2} 4\sigma^4 = \frac{N_0^2 W^2}{W^2} = N_0^2 \tag{12}$$

1.2 After Averaging

Since both averaging and correlating are linear operations, we can swap orders. Doing so I can now analyze averaging first:

$$z[n] = \frac{1}{N_{sync}} \sum_{k=1}^{N_{sync}} y[n + kN_b], \tag{13}$$

where N_b is the number of energy blocks per symbol and N_{sync} is the number of symbol repetition for the preamble. Knowing that despite the bandlimitation the considered ED samples are independent, we can say that

$$\mathrm{E}\{N_{sync}z[n]\} = N_{sync}\mathrm{E}\{y[i]\} \quad \rightarrow \quad \mathrm{E}\{z[n]\} = \frac{1}{N_{sync}} N_{sync}N_0 = N_0 \tag{14}$$

$$\mathrm{Var}\{N_{sync}z[n]\} = N_{sync}\mathrm{Var}\{y[i]\} \quad \rightarrow \quad \mathrm{Var}\{z[n]\} = \frac{1}{N_{sync}^2} N_{sync}N_0^2 = \frac{N_0^2}{N_{sync}} \tag{15}$$

1.3 After Correlation

Correlation is closely related to averaging, because also in this case we are summing-up different samples, which again can be assumed independent (this time because of the zero-spreading). Let $p[m]$ denote the power delay profile estimation we wish to obtain from the correlation:

$$p[m] = \frac{1}{N_{code}} \sum_{n=0}^{N_{reg}} R_i[n]z[n-m] \cong \sum_{n=1}^{N_N} z[n] - c \sum_{n=1}^{N_N-1} z[n], \tag{16}$$

where N_{reg} is the length of the register for correlating, N_{code} is the length of the code (and the number of non-zero samples in the reference symbol R_i) and N_N is the number of non-zero samples in the original symbol (usually $N_N = \frac{N_{code}+1}{2}$). c in this case denotes a constant which distinguishes between the code with zero-mean and the code with perfect autocorrelation properties:

$$c = \begin{cases} 1 & \text{perfect autocorrelation} \\ \frac{N_N}{N_N-1} & \text{zero-mean} \end{cases} \tag{17}$$

Taking into consideration that for independent samples the expected value of a sum is the sum of expected values we can simplify the calculation:

$$\mathrm{E}\{p[m]\} = \frac{1}{N_{code}} \left(\sum_{n=1}^{N_N} \mathrm{E}\{z[n]\} - c \sum_{n=1}^{N_N-1} \mathrm{E}\{z[n]\} \right) \tag{18}$$

$$= \frac{1}{N_{code}} \Big(N_N\mathrm{E}\{z[n]\} - c(N_N-1)\mathrm{E}\{z[n]\} \Big) = \frac{1}{N_{code}} \Big(N_N N_0 - c(N_N-1)N_0 \Big) \tag{19}$$

$$= \begin{cases} \frac{N_0}{N_{code}} & \text{perfect autocorrelation} \\ 0 & \text{zero-mean} \end{cases} \tag{20}$$

For the variance of the correlator outputs I will use the following relationship:

$$\mathrm{Var}\{p[m]\} = \mathrm{E}\left\{p^2[m]\right\} - \mathrm{E}\{p[m]\}^2 \tag{21}$$

where

$$\mathrm{E}\left\{N_{code}^2 p^2[m]\right\} = \mathrm{E}\left\{\Big(\sum_{n=1}^{N_N} z[n] - c\sum_{k=1}^{N_N-1} z[k]\Big)^2\right\} \tag{22}$$

$$= \mathrm{E}\left\{\Big(\sum_{n=1}^{N_N} z[n]\Big)^2\right\} - 2c\mathrm{E}\left\{\Big(\sum_{n=1}^{N_N} z[n]\sum_{k=1}^{N_N-1} z[k]\Big)\right\} + c^2\mathrm{E}\left\{\Big(\sum_{k=1}^{N_N-1} z[k]\Big)^2\right\} \tag{23}$$

$$\tag{24}$$

Using $\mu_z = \mathrm{E}\{z[n]\}$ and $\sigma_z^2 = \mathrm{Var}\{z[n]\}$ and the fact that $z[n]$ is independent, we can conclude:

$$\mathrm{E}\left\{\Big(\sum_{n=1}^{N_N} z[n]\Big)^2\right\} = N_N\mathrm{E}\left\{z^2[n]\right\} + N_N(N_N-1)\mathrm{E}\{z[n]z[k]\} \tag{25}$$

$$= N_N(\mu_z^2+\sigma_z^2) + N_N(N_N-1)\mu_z^2 = N_N\sigma_z^2 + N_N^2\mu_z^2 \tag{26}$$

$$\mathrm{E}\left\{\Big(\sum_{n=1}^{N_N} z[n]\sum_{k=1}^{N_N-1} z[k]\Big)\right\} = N_N(N_N-1)\mathrm{E}\{z[n]z[k]\} = N_N(N_N-1)\mu_z^2 \tag{27}$$

$$\mathrm{E}\left\{\Big(\sum_{k=1}^{N_N-1} z[k]\Big)^2\right\} = (N_N-1)\sigma_z^2 + (N_N-1)^2\mu_z^2 \tag{28}$$

And thus:

$$\mathrm{E}\left\{N_{code}^2 p^2[m]\right\} = N_N\sigma_z^2 + N_N^2\mu_z^2 - 2cN_N(N_N-1)\mu_z^2 + c^2(N_N-1)\sigma_z^2 + c^2(N_N-1)^2\mu_z^2 \tag{29}$$

$c = 1$

$$\mathrm{E}\left\{N_{code}^2 p^2[m]\right\} = N_N\sigma_z^2 + N_N^2\mu_z^2 - 2N_N(N_N-1)\mu_z^2 + (N_N-1)\sigma_z^2 + (N_N-1)^2\mu_z^2 \tag{30}$$

$$= \underbrace{(2N_N-1)}_{N_{code}}\sigma_z^2 + N_N^2\mu_z^2 - 2N_N^2\mu_z^2 + 2N_N\mu_z^2 + N_N^2\mu_z^2 - 2N_N\mu_z^2 + \mu_z^2 \tag{31}$$

$$= N_{code}\sigma_z^2 + \mu_z^2 \tag{32}$$

$$\mathrm{E}\left\{p^2[m]\right\} = \frac{\sigma_z^2}{N_{code}} + \frac{\mu_z^2}{N_{code}^2} = \frac{N_0^2}{N_{sync}N_{code}} + \frac{N_0^2}{N_{code}^2} \tag{33}$$

$$\mathrm{Var}\{p[m]\} = \frac{N_0^2}{N_{sync}N_{code}} = c\frac{N_0^2}{N_{sync}N_{code}} \tag{34}$$

$c = \frac{N_N}{N_N-1}$

$$\mathrm{E}\left\{N_{code}^2 p^2[m]\right\} = N_N\sigma_z^2 + N_N^2\mu_z^2 - 2\frac{N_N}{N_N-1}N_N(N_N-1)\mu_z^2 + \frac{N_N^2}{(N_N-1)^2}(N_N-1)\sigma_z^2 \tag{35}$$

$$+\frac{N_N^2}{(N_N-1)^2}(N_N-1)^2\mu_z^2 \tag{36}$$

$$= N_N\sigma_z^2 + N_N^2\mu_z^2 - 2N_N^2\mu_z^2 + \frac{N_N^2}{N_N-1}\sigma_z^2 + N_N^2\mu_z^2 \tag{37}$$

$$= \frac{N_N(2N_N-1)}{N_N-1}\sigma_z^2 = \frac{N_N}{N_N-1}N_{code}\sigma_z^2 = cN_{code}\sigma_z^2 \tag{38}$$

$$\mathrm{E}\left\{p^2[m]\right\} = c\frac{\sigma_z^2}{N_{code}} = c\frac{N_0^2}{N_{sync}N_{code}} \tag{39}$$

$$\mathrm{Var}\{p[m]\} = c\frac{N_0^2}{N_{sync}N_{code}} \tag{40}$$

That is, the variance is slightly higher for the zero-mean reference symbol.